

ON SOME PROBLEMS RELATED TO THE HILBERT-SMITH CONJECTURE

ALEXANDER DRANISHNIKOV

1. INTRODUCTION

The 5th Hilbert's problem [H] has an unsettled extension known as the Hilbert-Smith conjecture [Sm],[Wi]:

1.1. Conjecture (Hilbert-Smith Conjecture). *If a compact group G acts effectively (freely) on a connected manifold, then G is a Lie group.*

It is known that the Hilbert-Smith conjecture is equivalent to the question whether the group of p -adic integers A_p can act effectively (freely) on a manifold [Sm2].

The Conjecture is proven for n -dimensional manifolds with $n \leq 2$ [MZ2] and $n = 3$ [P]. For arbitrary n the Hilbert-Smith Conjecture is proven for smooth actions on a smooth manifold [MZ2], for Lipschitz actions on Riemannian manifolds [RSc], for Hölder actions [Mal], and for quasi-conformal actions [Mar].

A quite deep but not yet successful line of research on the Hilbert-Smith Conjecture is known as the orbit space method. For an effective action of a compact group G on a space X the formula for the dimension of the orbit space

$$\dim X/G = \dim X - \dim G$$

seemed quite natural. Since A_p is homeomorphic to the Cantor set and hence, $\dim A_p = 0$, one would expect the dimension of the orbit space M/A_p of an effective action of an n -manifold to be equal n . Contrary to that P.A. Smith found in 1940 [Sm] that this dimension is not n . Later C.T. Yang proved [Y1] that the cohomological dimension $\dim_{\mathbb{Z}} M/A_p$ of the orbit space of an effective p -adic action on an n -manifold M equals $n+2$. Therefore, by Alexandroff's theorem [A],[Wa] about the coincidence of cohomological and covering dimensions in the case when the latter is finite either $\dim M/A_p = n+2$ or $\dim M/A_p =$

Date: July 4, 2016.

2000 *Mathematics Subject Classification.* Primary 55M30; Secondary 53C23, 57N65 .

Supported by NSF, grant DMS-1304627.

∞ . The other surprising properties of the orbit spaces of a hypothetical p -adic action on a manifold can be found in [BRW], [R],[RW],[Wi],[Y2]. Still there is a remote hope that those bizarre properties of the orbit space M/A_p could lead to a contradiction and prove the Hilbert-Smith Conjecture.

In this paper we consider the Hilbert-Smith Conjecture under the assumption that the dimension of the orbit space is finite. We consider only free actions.

1.2. Conjecture (Weak Hilbert-Smith conjecture). *If a compact group G acts freely on a manifold M and dimension of the orbit space M/G is finite, then G is a Lie group.*

We reduce the weak Hilbert-Smith conjecture to two problems which we call the *Essential Lens Sequence* problem (Problem 3.1) and the *Injectivity Conjecture* (Conjecture 5.6). The reduction is based on the idea of Williams to use the infinite product in the K-theory of some special version of a classifying space BA_p for the group A_p [Wi]. We should warn the reader that in the definition of classifying spaces for A_p in [Wi] the order of direct and inverse limit must be exchanged.

The Essential Lens Sequence problem is a quest for compact 'classifying space' (we call it here a *rough classifying space*) which has an infinite product in K-theory and hence infinite dimensional. A finite dimensional compact rough classifying space for A_p was constructed by Floyd [Wi]. Existence of such an infinite dimensional rough classifying space together with Borel's construction would bring examples of cell-like maps which are Hurewicz fibrations and yet have nontrivial cokernels in K-theory. This looks surprising but it does not contradict to any known facts about cell-like maps. Thus, it is known that cell-like maps of manifolds can have nontrivial kernels in homology K-theory [DFW].

The Injectivity Conjecture is a technical statement about Hurewicz fibrations with the fiber a closed manifold M over a fixed base B . It states that for a fixed nonzero generalized cohomology class $\alpha \in h^*(B \times M)$ for a sufficiently close approximation $f : E \rightarrow B \times M$ of the trivial fibration $\pi : B \times M \rightarrow B$ by a Hurewicz fibration $p : E \rightarrow B$ with the fiber M the map f takes α to a nonzero element $f^*(\alpha)$. We apply the Injectivity Conjecture when h^* is the reduced K-theory. We note that for general spaces B the concept of Hurewicz fibration is a peculiar one, since B does not necessarily support interesting homotopies. There is a seemingly weaker notion of a completely regular map which does not appeal to homotopy in its definition. It is still unknown (the Hurewicz Fibration Problem) whether every completely regular map is

a Hurewicz fibration. We conclude the paper by reductions of the Injectivity Conjecture first to the Hurewicz Fibration Problem and then to the ANRness problem of the classifying space for completely regular maps with a given manifold fiber.

1.3. Theorem. *A positive solutions to both the Essential Lens Sequence Problem and the Hurewicz Fibration Problem imply the Weak Hilbert-Smith Conjecture for closed aspherical manifolds.*

1.4. Theorem. *If the Essential Lens Sequence Problem has positive solution and the classifying space for completely regular maps with a given compact Q -manifold is an absolute neighborhood extensor for compact metric spaces, then the Weak Hilbert-Smith Conjecture holds true.*

Our approach to the Hilbert-Smith Conjecture does not exclude a possibility of a p -adic action on a manifold with an infinite dimensional orbit space. On the other hand still there is no known examples of p -adic actions on a finite dimensional compact space with an infinite dimensional orbit space. Though there are examples of such actions of Cantor groups [DW],[Le].

2. PRELIMINARIES

2.1. p -Adic actions. By $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ we denote the cyclic group of order m . Let p be a prime number. The p -adic integers is a topological group defined as the inverse limit of the sequence

$$\mathbb{Z}_p \leftarrow \mathbb{Z}_{p^2} \leftarrow \mathbb{Z}_{p^3} \leftarrow \dots$$

where every bonding map $\mathbb{Z}_{p^{k+1}} \rightarrow \mathbb{Z}_{p^k}$ is the mod p^k reduction. Note that every closed subgroup of A_p has the form $p^k A_p$. If the group A_p acts on a compact metric space X , then X can be presented as the limit space of the inverse sequence

$$(*) \quad Y_0 \xleftarrow{q_0^1} Y_1 \xleftarrow{q_1^2} Y_2 \xleftarrow{q_2^3} Y_3 \xleftarrow{\quad} \dots$$

with $Y_0 = X/A_p$ and each space Y_k equals to the orbit space $X/p^k A_p$ of the action of the subgroup $p^k A_p$, all the bonding maps q_k^{k+1} are the projection to the orbit space of a \mathbb{Z}_p -action, and every composition

$$q_k^{k+i} = q_k^{k+1} \circ q_{k+1}^{k+2} \cdots \circ q_{k+i-1}^{k+i} : Y_{k+i} \rightarrow Y_k$$

is the projection onto the orbit space of an action of the quotient group $\mathbb{Z}_{p^i} = p^k A_p / p^{k+i} A_p$.

2.1. Proposition. *Suppose that the group of p -Adic integers A_p acts freely on a connected and locally connected compact metric space X with the orbit space Y . Then X and Y can be presented as the inverse limit*

of sequences of simplicial complexes such that there is a commutative diagram

$$\begin{array}{ccccccccc}
 K_0 & \xleftarrow{\phi_0^1} & K_1 & \xleftarrow{\phi_1^2} & K_2 & \xleftarrow{\phi_2^3} & K_3 & \xleftarrow{\quad} & \cdots & X \\
 p_0 \downarrow & & p_1 \downarrow & & p_2 \downarrow & & p_3 \downarrow & & & q_0 \downarrow \\
 L_0 & \xleftarrow{\psi_0^1} & L_1 & \xleftarrow{\psi_1^2} & L_2 & \xleftarrow{\psi_2^3} & L_3 & \xleftarrow{\quad} & \cdots & Y
 \end{array}$$

where each p_k is the projection onto the orbit space of a free action of \mathbb{Z}_{p^k} , each bonding map ϕ_{k-1}^k is \mathbb{Z}_{p^k} -equivariant.

Proof. We take $Y = Y_0$ from (*) and construct the inverse sequence $\{L_i, \psi_{i-1}^i\}$ using nerves of a sequence of finite open covers $\{\mathcal{U}_i\}$ with $\text{mesh } \mathcal{U}_i \rightarrow 0$. Since Y is locally connected, we may assume that all sets in each \mathcal{U}_i are connected. Thus, $L_i = \text{Nerve}(\mathcal{U}_i)$. Let $\psi_i : Y \rightarrow L_i$ denote the projection to the nerve. We recall that it is defined by means of a partition of unity subordinated by \mathcal{U}_i .

We may assume that each $U \in \mathcal{U}_i$ admits a section of q_0^i . Thus, the preimage $(q_0^i)^{-1}(U)$ is a disjoint union of p^i copies of U . These copies of U define a finite open cover \mathcal{V}_i of Y_i . Let $K_i = \text{Nerve}(\mathcal{V}_i)$. Note that there is a simplicial map $q_i : K_i \rightarrow L_i$ which is the projection onto the orbit space of \mathbb{Z}_{p^i} -action. Moreover, there is a map $\psi'_i : Y_i \rightarrow K_i$ which defines a pull-back diagram

$$\begin{array}{ccc}
 K_i & \xleftarrow{\psi'_i} & Y_i \\
 p_i \downarrow & & q_0^i \downarrow \\
 L_i & \xleftarrow{\psi_i} & Y.
 \end{array}$$

It is an easy exercise to show that the multi-valued upper semi-continuous map $F : K_i \rightarrow K_{i-1}$ defined by the formula $F(x) = \psi'_{i-1}(\psi'_i)^{-1}$ is in fact single-valued. Thus F defines a continuous map ϕ_{i-1}^i . We show that the square diagram in our sequence commutes $p_{i-1}\phi_{i-1}^i = \psi_{i-1}^i p_i$:

$$\begin{aligned}
 p_{i-1}\phi_{i-1}^i &= p_{i-1}F(x) = p_{i-1}\psi'_{i-1}((\psi'_i)^{-1}(x)) = \psi_{i-1}q_0^i((\psi'_i)^{-1}(x)) \\
 &= \psi_{i-1}(\psi_i^{-1}(p_i(x))) = \psi_{i-1}^i p_i(x).
 \end{aligned}$$

Clearly, $\lim_{\leftarrow} \{K_i, \phi_{i-1}^i\} = X$. □

2.2. Borel construction. Let a group G act on spaces X and E with the projections onto the orbit spaces $q_X : X \rightarrow X/G$ and $q_E : E \rightarrow E/G$. Let $q_{X \times E} : X \times E \rightarrow X \times_G X = (X \times E)/G$ denote the projection

to the orbit space of the diagonal action of G on $X \times E$. Then there is a commutative diagram called the *Borel construction* [AB]:

$$\begin{array}{ccccc} X & \xleftarrow{pr_X} & X \times E & \xrightarrow{pr_E} & E \\ q_X \downarrow & & q_{X \times E} \downarrow & & q_E \downarrow \\ X/G & \xleftarrow{p_E} & X \times_G E & \xrightarrow{p_X} & E/G. \end{array}$$

If G is compact, the actions are free, and q_E is locally trivial bundle, then so is p_X . In particular, this holds true for free actions of compact Lie groups. Since the projection of the limit space of an inverse sequence whose bonding maps are locally trivial fibrations onto the first space of the sequence is a Hurewicz fibration, using approximation of a compact group by compact Lie groups, we obtain that in the case of free G -actions all projections in the diagram are Hurewicz fibrations. The fiber $p_X^{-1}(y)$ is homeomorphic to X/I_z where $I_z = \{g \in G \mid g(z) = z\}$ is the isotropy group of $z \in q_E^{-1}(y)$.

3. ESSENTIAL SEQUENCES OF LENS SPACES

For an integer m we denote the standard $(2n - 1)$ -dimensional lens space mod m by

$$L^n(m) = S^{2n-1}/\mathbb{Z}_m$$

where the \mathbb{Z}_m -action on $(2n - 1)$ -sphere $S^{2n-1} \subset \mathbb{C}^n$ is obtained from the rotation by $2\pi/m$ in every coordinate plane \mathbb{C} in \mathbb{C}^n .

Note that the classifying space $B\mathbb{Z}_m = K(\mathbb{Z}_m, 1)$ can be presented as the increasing union $\cup_n L^n(m)$.

We call a map between lens spaces $q : L^m(p^k) \rightarrow L^n(p^\ell)$ *essential* if it induces an epimorphism of the fundamental groups. A sequence of mappings of spheres

$$S^{k_0} \xleftarrow{f_0^1} S^{k_1} \xleftarrow{f_1^2} S^{k_2} \xleftarrow{f_2^3} \dots$$

is called *inessential* if for every i there is j such the $f_i^{i+1} \circ \dots \circ f_{i+j-1}^{i+j}$ is null-homotopic.

3.1. Problem (Essential Lens Sequence Problem). *Given an odd prime p , does there exist an infinite sequence of lens spaces*

$$(1) \quad L^{n_0}(p^{k_0}) \xleftarrow{q_0^1} L^{n_1}(p^{k_1}) \xleftarrow{q_1^2} L^{n_2}(p^{k_2}) \xleftarrow{q_2^3} \dots$$

with essential bonding maps, $k_i \rightarrow \infty$, and $n_i > p^{k_i - k_0}$, such that the sequence of covering spheres

$$(2) \quad S^{2n_0-1} \xleftarrow{\bar{q}_0^1} S^{2n_1-1} \xleftarrow{\bar{q}_1^2} S^{2n_2-1} \xleftarrow{\bar{q}_2^3} \dots$$

is inessential ?

It is known (see Section 4) that $d(c)$, the dimension of a lens space in an essential sequence as a function of the cardinality of its fundamental group, can grow at most linearly. Floyd's example [Wi] gives us an essential sequence of lens spaces with constant function $d(c) = 3$. Using ideas from [Dr] it is possible to construct an essential sequence with $d(c) \sim \log c$. It turns out that for our applications to the Hilbert-Smith conjecture we need $d(c)$ to be linear. This requirement is spelled out in the condition: $n_i > p^{k_i - k_0}$.

The ELS problem can be stated for concrete values of k_i and n_i :

- (1) $k_i = i + 1$ and $n_i = p^i + 1$;
- (2) $k_i = i + 1$ and $n_i = p^i + 2$;
- (3) $k_i = 2^i$ and $n_i = p^{2^i - 1} + 1$.

Suppose that a sequence from Problem 3.1 does exist. Then the maps $q^{i+1} : L^{n_{i+1}}(p^{k_{i+1}}) \rightarrow L^{n_i}(p^{k_i})$ define maps $\tilde{q}_i^{i+1} : S^{2n_{i+1}-1} \rightarrow S^{2n_i-1}$ between the universal covers. Denote by

$$E = \varprojlim \{S^{2n_i-1}, \tilde{q}_i^{i+1}\}$$

and by

$$B = \varprojlim \{L^{n_i}(p^{k_i}), q_i^{i+1}\}$$

the corresponding inverse limits. Then the following proposition is straightforward.

3.2. Proposition. *The compactum E is cell-like and there is a free A_p -action on E with the orbit space B .*

Using Ferry's theorem [F1] we can modify the bonding maps (possibly with stabilizations) in the inverse sequence $\varprojlim \{L^{n_i}(p^{k_i}), q_i^{i+1}\}$ to UV^0 -maps. Then we may assume that B and E are path connected and locally path connected.

3.1. K-theory of lens spaces. For any r the actions of the groups $\mathbb{Z}_{p^r} \subset \mathbb{Z}_{p^{r+1}} \subset S^1$ on S^∞ and S^{2n-1} form a commutative diagram

$$\begin{array}{ccccccc} S^\infty & \longrightarrow & B\mathbb{Z}_{p^k} & \longrightarrow & B\mathbb{Z}_{p^{k+1}} & \longrightarrow & \mathbb{C}P^\infty \\ \subset \uparrow & & \subset \uparrow & & \subset \uparrow & & \subset \uparrow \\ S^{2n-1} & \longrightarrow & L^n(p^r) & \longrightarrow & L^n(p^{r+1}) & \longrightarrow & \mathbb{C}P^n. \end{array}$$

The canonical line bundle η over $\mathbb{C}P^\infty$ defines the line bundles over all spaces in the diagram. This bundle defines an element in the K-theory of the Eilenberg-MacLane space $B\mathbb{Z}_{p^k}$ which will be denoted by η_k .

As it was computed by Atiyah (see [AS]) the K-theory of $B\mathbb{Z}_{p^k}$ equals the completion of the representation ring $R\mathbb{Z}_{p^k}$ of \mathbb{Z}_{p^k} . We recall that,

$R\mathbb{Z}_{p^k} = \mathbb{Z}[\eta]/\langle 1 - \eta^{p^k} \rangle$ where η is the class of the complex representation of \mathbb{Z}_{p^k} generated by the group embedding $\eta : \mathbb{Z}_{p^k} \rightarrow S^1$. Note that η_k is obtained from η by passing to the map of classifying spaces $\eta : B\mathbb{Z}_{p^k} \rightarrow BS^1 = \mathbb{C}P^\infty$ and pulling back the canonical complex line bundle. Thus, taking the completion we obtain

$$K^0(B\mathbb{Z}_{p^k}) = (R\mathbb{Z}_{p^k})^\wedge = \mathbb{Z}[[\eta_k]]/\langle 1 - \eta_k^{p^k} \rangle$$

where $A[[x]]$ denotes the ring of formal series with the variable x and coefficients in A . Note that the mod p^k reduction homomorphism $\mathbb{Z}_{p^{k+r}} \rightarrow \mathbb{Z}_{p^k}$ takes the generator η_k to $\eta_{k+r}^{p^r}$.

Let $\bar{\eta}_k$ denote the restrictions of these classes to the $(2n-1)$ -dimensional lens space $L^n(p^k)$. The K -theory of this lens space was computed in [Ma], [Ka]:

$$K^0(L^n(p^k)) = \mathbb{Z}[\bar{\eta}_k]/\langle 1 - \bar{\eta}_k^{p^k}, (\bar{\eta}_k - 1)^n \rangle.$$

We note that the ideal generated by $\bar{\eta}_k - 1$ in the above ring is isomorphic to the reduced K -theory of $L^n(p^k)$ (see [KS]).

3.3. Proposition. *For any positive integers $k > l$, $m < p^l$, and prime p the polynomial $(x^{p^{k-l}} - 1)^m$ does not belong to the ideal $\langle x^{p^k} - 1, (x-1)^n \rangle$ of the polynomial ring $\mathbb{Z}[x]$ provided that $mp^{k-l} < n$.*

Proof. We change the variable $y = x - 1$. Thus, we need to show that $((y+1)^{p^{k-l}} - 1)^m$ does not belong to $\langle (y+1)^{p^k} - 1, y^n \rangle$. The mod p^{k-l} reduction of this problem is the question whether $y^{mp^{k-l}}$ belongs to the ideal $\langle y^{p^k}, y^n \rangle$. Since $mp^{k-l} < \min\{p^k, n\}$, the answer to the question is negative and the result follows. \square

3.4. Proposition. *Let $q_i : B \rightarrow L^{n_i}(p^{k_i})$ be the projection of the limit in the above inverse system. Suppose that $m < p^{k_i - k_0}$. Then the induced homomorphism in the reduced K -theory*

$$q_i^* : \tilde{K}^0(L^{n_i}(p^{k_i})) \rightarrow \tilde{K}^0(B)$$

takes $(\bar{\eta}_{k_i} - 1)^m$ to a nonzero element.

Proof. Note that

$$(q_i^{i+j})^*((\bar{\eta}_{k_i} - 1)^m) = (\bar{\eta}_{k_{i+j}}^{p^{k_{i+j} - k_i}} - 1)^m.$$

We apply Proposition 3.3 with $k = k_{i+j}$, $l = k_i$, $n = n_{i+j}$ to obtain that $(\bar{\eta}_{k_{i+j}}^{p^{k_{i+j} - k_i}} - 1)^m \neq 0$ since $m < p^{k_i}$ and

$$mp^{k_{i+j} - k_i} < p^{k_i - k_0} p^{k_{i+j} - k_i} = p^{k_{i+j} - k_0} < n_{i+j},$$

i.e., the conditions of Proposition 3.3 are satisfied. \square

3.5. Corollary. *The reduced K -theory cup length of B is unbounded.*

4. LENS SEQUENCE PROBLEM AND THE SCHWARZ GENUS

We consider the level functions defined in [Me]

$$v_{p,k}(m) := \min\{n \mid \exists f : L^m(p^{k-1}) \xrightarrow{\mathbb{Z}_p} S^{2n-1}\}$$

where f is a \mathbb{Z}_p -equivariant map with respect to the standard free actions. Lower bounds for these functions were given by Vick [V], rediscovered by Bartsch [Ba], and formulated in this way by D. Meyer [Me]:

$$v_{p,k}(m) \geq \lceil \frac{m-1}{p^{k-1}} \rceil + 1.$$

Also D. Meyer has computed $v_{p,2}(m) = \frac{m-2}{p} + 2$ for odd p and $m = 2 \bmod p$ [Me].

REMARK. The existence of a sequence of lens spaces as in Problem 3.1 does not contradict to the above estimates of the level functions. Indeed, for $i > j$ the composition $q_j^{j+1} \circ \dots \circ q_{i-1}^i$ induces a \mathbb{Z}_p -equivariant map $L^{n_i}(p^{k_i-k_j}) \rightarrow S^{2n_j-1}$. Therefore, we have the inequality

$$n_j \geq v_{p,k_i-k_j+1}(n_i) \geq \lceil \frac{n_i-1}{p^{k_i-k_j}} \rceil + 1 \geq \frac{p^{k_i-k_0}}{p^{k_i-k_j}} + 1 > p^{k_j-k_0}$$

which is consistent with the conditions on k_i and n_i .

4.1. Schwarz genus. We recall that the Schwarz genus $Sg(f)$ of a fibration $f : E \rightarrow B$ is the minimal k such that B can be covered by open sets A_1, \dots, A_k such that p admits a section on each A_i [Sch].

4.1. Proposition. [Sch] $Sg(f) \leq n$ if and only if $*^n f : *^n_B E \rightarrow B$ admits a section.

A free action of \mathbb{Z}_r on S^1 determines a free \mathbb{Z}_r -action on $S^{2m-1} = *^m S^1$, and a free action of \mathbb{Z}_{pr} on S^{2m-1} determines a free \mathbb{Z}_r -action on $L^m(p)$. In the question below we consider the free \mathbb{Z}_{p^k} -actions on $L^m(p)$ and S^1 determined in this way.

Let $\pi_k^m = p_{S^1} : W_k^m = L^m(p) \times_{\mathbb{Z}_{p^k}} S^1 \rightarrow L^m(p^{k+1})$ be the S^1 -bundle from the Borel construction for \mathbb{Z}_{p^k} -actions on S^1 and $L^m(p)$.

4.2. Theorem. *There is a map $q : L^m(p^{k+1}) \rightarrow L^n(p^k)$ that induces an epimorphism of the fundamental groups if and only if $Sg(\pi_k^m) \leq n$.*

Proof. Such map q exists if and only if there is a \mathbb{Z}_{p^k} -equivariant map $q' : L^m(p) \rightarrow S^{2n-1}$ for free actions. This is equivalent to the existence of a section of the locally trivial S^{2n-1} -bundle $\pi : L^m(p) \times_{\mathbb{Z}_{p^k}} S^{2n-1} \rightarrow L^m(p^{k+1})$ from the Borel construction. Since the sphere $S^{2n-1} = *_{i=1}^n S^1$

is the join product of n circles and the action comes from a free \mathbb{Z}_{p^k} -action on S^1 , the bundle π is the fiberwise join product of n copies of the S^1 -bundle π_k^m . Then the result follows from Proposition 4.1. \square

4.3. Corollary. $Sg(\pi_k^m) \geq \lceil \frac{m-1}{p} \rceil + 1$.

For odd p , $Sg(\pi_1^m) = \frac{m-2}{p} + 2$ provided $m = 2 \bmod p$.

Proof. Suppose that $Sg(\pi_k^m) = n$. Then there is a map $q : L^m(p^{k+1}) \rightarrow L^n(p^k)$ that induces an epimorphism of the fundamental groups. Going to \mathbb{Z}_{p^k} covers we obtain a \mathbb{Z}_{p^k} -equivariant map $f : L^m(p) \rightarrow S^{2n-1}$. Thus, the map f is \mathbb{Z}_p -equivariant as well, $\mathbb{Z}_p \subset \mathbb{Z}_{p^k}$. Hence $v_{p,2}(m) \leq n$. We proved the inequality $Sg(\pi_k^m) \geq v_{p,2}(m)$. The cited above results of Vick, Bartsch, and D. Meyer imply $Sg(\pi_k^m) \geq \lceil \frac{m-1}{p} \rceil + 1$. \square

4.4. Problem. (1) What is the Schwarz genus of the S^1 -bundle from the Borel construction $\pi_k^m : L^m(p) \times_{\mathbb{Z}_{p^k}} S^1 \rightarrow L^m(p^{k+1})$?

(2) In particular, if $m = 2 \bmod p$, is $Sg(\pi_k^m) = \frac{m-2}{p} + 2$ for all k ?

4.5. Corollary. A positive answer to Problem 4.4 (2) gives a positive answer to Problem 3.1.

Proof. We take $k_i = i$ and $n_i = 3p^i - \sum_{s=1}^{i-1} p^s + 1$. Clearly, the condition $n_i > p^{k_i - k_0}$ is satisfied. Note that $m = n_{i+1} + 1$ equals $2 \bmod p$ and $\frac{m-2}{p} + 2 = 3p^i - \sum_{s=1}^{i-1} p^s - 1 \leq n_i$. If $Sg(\pi_i^m) = \frac{m-2}{p} + 2$, then by Theorem 4.2 there is an essential map $f_i^{i+1} : L^m(p^{i+1}) \rightarrow L^{n_i}(p^i)$. We define $q_i^{i+1} : L^{n_{i+1}}(p^{i+1}) \rightarrow L^{n_i}(p^i)$ to be the restriction of f_i^{i+1} to $L^{n_{i+1}}(p^{i+1}) \subset L^m(p^{i+1})$. Then the corresponding sequence of spheres will be inessential. \square

5. A REDUCTION OF THE WEAK HILBERT-SMITH CONJECTURE

5.1. Dimension, LS-category, and cup-length. We recall that a topological space X has the Lusternick-Schnirelmann category (LS-category for short) $cat(X) \leq n$ if there is an open cover U_0, \dots, U_n of X by $n+1$ contractible in X sets. It's known that in the case of ANR, it suffices to take closed U_i s or even arbitrary [Sr]. We refer to [CLOT] for general properties of the LS-category. The basic properties are that $cat(X)$ is a homotopy invariant, it is bounded from above by dimension $\dim X$, and from below by the length of a nonzero cup product of nonzero dimensional elements in cohomology. It is known that the cohomology in this cup product could be generalized or even 0-dimensional if one uses a reduced cohomology theory defined by means of a spectrum [Sw].

5.1. Proposition. *The LS-category of a finite connected complex is greater than or equal to the cup-length for any reduced generalized cohomology theory \tilde{h}^* .*

Proof. This proposition can be extracted from [Ru]. We give an alternative proof, since it is short. Assume that $w = \alpha_1 \smile \cdots \smile \alpha_k \neq 0$ in $\tilde{h}^*(X)$. Suppose that $\text{cat}_{\text{LS}} X \leq k - 1$. Then there is a cover U_1, \dots, U_k of X by contractible in X subcomplexes (for some subdivision). The long exact sequence of pair (X, U_i) for the reduced h -cohomology and the fact that $\tilde{h}^*(X) \rightarrow \tilde{h}^*(U_i)$ are 0-homomorphisms imply that the homomorphisms $\tilde{h}^*(X; U_i) \rightarrow \tilde{h}^*(X)$ are isomorphisms in all dimension. Let $\bar{\alpha}_i$ denote the corresponding elements in $\tilde{h}^*(X; U_i)$. Then the product $\bar{w} = \bar{\alpha}_1 \smile \cdots \smile \bar{\alpha}_k \neq 0$. On the other hand, $\bar{w} \in \tilde{h}^*(X; U_1 \cup \cdots \cup U_k) = \tilde{h}^*(X; X) = 0$. We have a contradiction. \square

5.2. Corollary. *The cup-length of a finite connected complex for any generalized reduced cohomology theory \tilde{h}^* does not exceed the dimension of the complex.*

5.3. Theorem. *For a finite dimensional compact metric connected space X , the cup-length for any reduced generalized cohomology theory does not exceed $\dim X$.*

Proof. Let $\dim X = n$. Then X can be presented as the inverse limit of a sequence of n -dimensional polyhedra $X = \lim_{\leftarrow} L_m$. If

$$\alpha_1 \smile \cdots \smile \alpha_k \neq 0$$

in $\tilde{h}^*(X)$, then there is m such that $\alpha_i = p_m^*(\beta_i)$, $i = 1, \dots, k$, and $\beta_1 \smile \cdots \smile \beta_k \neq 0$ where $p_k : X \rightarrow L_k$ is the projection in the inverse system. By Corollary 5.2, $k \leq n$. \square

5.2. Injectivity conjecture. The Chapman-Ferry α -approximation theorem [CF2] has severer versions. For instance Theorems 1, 2, 3, 4 in [F3] are all variations of that. One of the weakest version states that for a fixed metric on a closed manifold M for every $\delta > 0$ there is $\epsilon > 0$ such that every ϵ -map $f : M \rightarrow M$ is δ -homotopy equivalence. The following conjecture is a parametrized version of this version of the α -approximation theorem with a fixed space of parameters B . Since we don't assume that B is nice, we replace in our conjecture the homotopy equivalence by a shape equivalence.

Let F be a compact metric space. We say that a fibration $p : E \rightarrow B$ is a *fibration with isometric fibers* F if there is a metric on E such that all fibers $p^{-1}(x)$, $x \in B$, are isometric to F .

5.4. Conjecture (Parametrized α -Approximation Conjecture). *For every compact manifold M (or Q -manifold) with a fixed metric on it and any connected and locally connected compact space B there is $\epsilon > 0$ such that for any Hurewicz fibration $p : E \rightarrow B$ with isometric fibers M every fiberwise ϵ -map $f : E \rightarrow M \times B$ is a shape equivalence. In particular, it induces an isomorphism of generalized cohomology groups.*

The α -Approximation Conjecture seems out of reach. For applications to the Hilbert-Smith conjecture it suffices to prove the following.

5.5. Conjecture (Injectivity Conjecture). *For every closed manifold M (or Q -manifold) with a fixed metric on it, any connected and locally connected compact space B , and any nonzero element $\alpha \in h^*(B)$ for some generalized cohomology theory h^* there is $\epsilon > 0$ such that for every Hurewicz fibration $p : E \rightarrow B$ with isometric fibers M and every fiberwise ϵ -map $f : E \rightarrow M \times B$, the image $f^*\pi^*(\alpha) \neq 0$ where $\pi : M \times B \rightarrow B$ is the projection.*

We note that the α -approximation theorem holds true for Hilbert cube manifolds (Q -manifolds). Thus, it makes sense to extend the Injectivity Conjecture to Q -manifolds as well. We note that the Injectivity Conjecture for Q -manifolds implies the Injectivity Conjecture for ordinary manifolds via multiplication by the Hilbert cube.

Let G be a compact metrizable topological group. The orbit space of a free G -action on a Peano continuum of a trivial shape will be called a *rough classifying space* for G .

5.6. Corollary (Corollary of the Injectivity Conjecture). *Let B be a rough classifying space for A_p . Suppose that A_p acts on a compact manifold (or Q -manifold) M . Then for any nonzero $\alpha \in K^0(B)$ there is k such that $p_M^*(\alpha) \neq 0$ where $p_M : M \times_{A_p} E \rightarrow B$ is the projection from the Borel construction for the action on M of the subgroup $p^k A_p \cong A_p$.*

Proof. Let μ be an invariant measure on A_p . Integration on A_p with respect to μ of a metric d on M gives an A_p -invariant metric on M : $\rho(x, y) = \int_{A_p} d(gx, gy) d\mu$. We take any metric d' on E and consider the ℓ_1 -product metric $\rho + d'$ on $M \times E$. This defines the quotient metric on each of the orbit spaces $E_k = M \times_{A_p} E$ for the diagonal action of A_p where for the action on M the group A_p is identified with $p^k A_p$. Thus, $p_M : E_k \rightarrow B$ is a fibration with isometric fibers M .

We claim that for large k the total space $E_k = M \times_{A_p} E$ of the Borel construction of the action of $p^k A_p$ on M admits a retraction onto a fiber M the restriction of which to any other fiber is an ϵ_k -map with

$\epsilon_k \rightarrow 0$. There are several ways to argue for this. We leave the proof to the reader. One approach would be that the composition of the inverse to the quotient map $M \times E \rightarrow E_k$ followed by a retraction $r : M \times E \rightarrow M$ to a fiber defines a multivalued such retraction with the diameter d_k of the images of points tending to 0. That with fact that M is ANR would be sufficient to derive our claim.

Then the Injectivity Conjecture implies the required result. \square

The following is the main result of the paper:

5.7. Theorem. *Suppose that the Injectivity Conjecture holds true and there exists an infinite sequence of lens spaces as in the Essential Lens Sequence problem. Then there is no free A_p -action on a closed manifold with a finite dimensional orbit space.*

Proof. Assume that there is such a free A_p -action on an n -manifold M with $\dim M/A_p < \infty$. Then by the Yang's theorem $\dim M/A_p = n + 2$. Let B and E be as in Proposition 3.2. We apply the Injectivity Conjecture to the product of a desired length

$$\alpha = \alpha_1 \cdots \alpha_{n+3} \in \tilde{K}^0(B)$$

defined by Corollary 3.5. Then $p_M^*(\alpha) \neq 0$ for the action of $p^k A_p$ for some k .

By Proposition 3.2 the projection p_E from the Borel construction for that action is a cell-like map. Since $\dim M/p^k A_p < \infty$, it is a shape equivalence [La] and hence it induces an isomorphism in K -theory. Hence for each i there is $\beta_i \in K^0(M/p^k A_p)$ such that $p_E^*(\beta_i) = p_M^*(\alpha_i)$. Hence, $\beta_1 \dots \beta_{n+3} \neq 0$.

$$\begin{array}{ccccc} M & \xleftarrow{pr_{\mathbb{R}^n}} & M \times E & \xrightarrow{pr_E} & E \\ q_M \downarrow & & q_{M \times E} \downarrow & & q_E \downarrow \\ M/p^k A_p & \xleftarrow{p_E} & M \times_{A_p} E & \xrightarrow{p_M} & B. \end{array}$$

Since $\dim M/(p^k A_p) = n + 2$, we obtain a contradiction with Theorem 5.3. \square

6. HUREWIZ FIBRATION PROBLEM

6.1. Completely regular maps. Dyer and Hamstrom introduced the notion of a completely regular map in [DH]. We recall that a continuous surjection $p : E \rightarrow B$ between metric spaces is called *completely regular* if for each $b \in B$ and $\epsilon > 0$, there exists $\delta(b, \epsilon) > 0$ such that if $d_B(b, b') < \delta$, then there exists a homeomorphism $h : p^{-1}(b) \rightarrow p^{-1}(b')$

with $d_E(x, h(x)) < \epsilon$ for all $x \in p^{-1}(b)$. It is known that the complete regularity of a map does not depend on choice of metrics d_B and d_E .

Using Michael's selection theorem [M], Dyer and Hamstrom proved the following theorem [DH].

6.1. Theorem. *Suppose that for a compact F the space $\text{Homeo}(F)$ is locally contractible. Then every completely regular map $p : E \rightarrow B$ with fiber F and a finite-dimensional B is a locally trivial fibration.*

The mistake in the proof presented in [DH] was corrected in [Ha]. A detailed proof can be found in [RS]. Similar or stronger related results later were proven in [K], [Se], [CF], [F2].

The Ferry's α -approximation theorem (Theorem 1 from [F3]) admits the following variation:

6.2. Theorem. *Let M be a closed n -manifold, $n \geq 5$, with a fixed metric. Then for any $\epsilon > 0$ there is $\delta > 0$ such that for every δ -map $g : M \rightarrow N$ onto a closed n -manifold N there is a homeomorphism $h : N \rightarrow M$ such that the composition $h \circ g$ is ϵ -close to the identity 1_M .*

Like the proof of the original Ferry's theorem, the proof of this variation is a consecutive applications of Theorems 2, 3, and 4 of [F3]. Also we note that this theorem holds true for compact Q -manifolds.

We recall that a map $f : X \rightarrow Y$ of a subset X of a metric space (Y, d) is called an ϵ -move if $d(x, f(x)) < \epsilon$ for all $x \in X$.

6.3. Proposition. *Let M be a close manifold with a fixed metric. Then given $\epsilon_0 > 0$, there is $\delta_0 > 0$ such that for every isometric embedding $M \subset X$ for any n -dimensional compact $Z \subset N_{\delta_0}(M)$ of the δ_0 -neighborhood of M there is continuous ϵ_0 -move $r : Z \rightarrow M$.*

Proof. By the Lefschetz criterion of ANRs (see [Bo], Theorem 8.1), for any $\epsilon > 0$ there is $\delta > 0$ such that for every map of the vertices $f : K^{(0)} \rightarrow M$ of a n -dimensional simplicial complex K with the condition that $d(f(v), (v')) < \delta$ for every edge $[v, v'] \subset K$ there is an extension $\bar{f} : K \rightarrow M$ with $\text{diam} \bar{f}(\Delta) < \epsilon$ for each simplex $\Delta \subset K$.

We prove the Proposition when Z is a polyhedron. For the general case (not needed in the paper) can be obtained using approximation of Z by nerves of small open covers.

We take $\epsilon < \epsilon_0/3$ to obtain $\delta < \epsilon$ from the Lefschetz criterion. Take $\delta_0 < \delta/4$ and consider a triangulation of Z with the mesh δ' satisfying $\delta' < \delta - 2\delta_0$. We may assume that $\delta' < \epsilon_0 - \epsilon - \delta_0$. We define r on the vertices of Z by sending each vertex v to a nearest point of M . Thus, $d(v, r(v)) < \delta_0$. Then for any edge $[v, v']$ we obtain $d(r(v), r(v')) <$

$2\delta_0 + \delta' < \delta$. Let $r : Z \rightarrow M$ be an extension from the Lefschetz criterion. Then for every $z \in Z$ we consider a simplex $\Delta \subset Z$ that contains z and fix a vertex $v \in \Delta$. By the triangle inequality we obtain $d(z, r(z)) \leq d(z, v) + d(v, r(v)) + d(r(v), r(z)) < \delta' + \delta_0 + \epsilon < \epsilon_0$. \square

6.4. Theorem. *Let $\phi : X \rightarrow Y$ be a continuous map between compact metric spaces such that all point preimages $\phi^{-1}(y)$ are isometric to a closed n -manifold M , $n \geq 5$. Then ϕ is a completely regular map.*

Proof. Let $y \in Y$ and $\epsilon > 0$ be given. Proposition 6.3 implies that there is a neighborhood $U(y)$ of $y \in Y$ such that for every $y' \in U(y)$ there is a $\delta/2$ -move $r : \phi^{-1}(y') \rightarrow \phi^{-1}(y)$ where δ is taken for $\epsilon/2$ as in Theorem 6.2. Let $r' : \phi^{-1}(y) \rightarrow \phi^{-1}(y')$ be a similar map back. We may assume that $r' \circ r$ is homotopic to the identity. Therefore, we may assume that r is surjective. Let $i : M \rightarrow \phi^{-1}(y')$ be an isometry. Note that $r \circ i$ is a δ -map. Hence there is a homeomorphism $h : \phi^{-1}(y) \rightarrow M$ with $d_M(hri(z), z) < \epsilon/2$. Note that the homeomorphism $i \circ h$ is an ϵ -move: $d_X(ih(x), x) = d_X(ihri(z), ri(z)) \leq d_X(ihri(z), i(z)) + d_X(i(z), ri(z)) = d_M(hri(z), z) + d_X(i(z), ri(z)) < \epsilon/2 + \delta/2 < \epsilon$. Here $z \in M$ is such that $ri(z) = x$. Such z exists in view of surjectivity of r . \square

We note that Theorem 6.4 holds true for Q -manifolds as well.

6.5. Question (The Hurewicz Fibration Problem). Is every completely regular map with a manifold fiber a Hurewicz fibration?

In view of Theorem 6.1, it is an open problem when the base is infinite dimensional. It is known that a completely regular map is a Serre fibration. We refer to [DS] for further discussion of the Fibration Problem.

6.2. Completely regular maps in the Borel construction. Suppose that a compact group G acts freely on a metric space E with the orbit space B . Suppose that it also acts on a compact space F . We may assume that G acts by isometries.

6.6. Proposition. *The projection $p_F : F \times_G E \rightarrow B$ in the Borel construction is completely regular.*

Proof. Fix $y \in q_E^{-1}(x)$. Since q_E is open, any sequence x_n converging to x in B admits a lift y_n converging to y in E with respect to the orbit map $q_E : E \rightarrow B$. Then

$$(q_{F \times E})(1_F \times c_i)((q_{F \times E})|_{F \times \{y\}})^{-1} : p_F^{-1}(x) \rightarrow p_F^{-1}(x_k)$$

is the sequence of homeomorphisms that converges to the identity $id : p_F^{-1}(x) \rightarrow p_F^{-1}(x)$ where $c_k : y \rightarrow y_k$ is the map of one-point spaces and $q_{F \times E} : F \times E \rightarrow F \times_G E$ is the orbit map of the diagonal action. \square

We use the notation $Homeo(M)$ for the group of homeomorphisms of a manifold M with the compact-open topology. By $Homeo_0(M)$ we denote the subgroup of homeomorphisms isotopic to the identity. For manifolds with boundary we use the notation $Homeo(M, \partial M)$ for the group of the homeomorphisms of M which are the identity on ∂M .

6.7. Question. Let M be a manifold and $G \subset Homeo_0(M)$ be a compact subgroup that admits a deformation $H : G \times [0, 1] \rightarrow Homeo(M)$ to the unit 1. Does there exist such a deformation H through homomorphisms, i.e., such that $h_t = H(-, t) : G \rightarrow Homeo(M)$ is a group homomorphism for every t ?

6.8. Theorem. *Suppose that Questions 6.5 and 6.7 have positive answers. Then the Injectivity Conjecture holds true.*

Proof. Let A_p act freely on a manifold F (or Q -manifold). Let h_t be a deformation of A_p to the identity in $Homeo(F)$ by virtue of a family of subgroups $h_t(A_p) \subset Homeo(M)$. We define an A_p -action on $F \times [0, 1]$ by letting the group $h_t(A_p)$ act on $F \times \{t\}$. Let B be a rough classifying space for A_p with the universal covering $q_E : E \rightarrow B$.

The projection $p_{F \times [0, 1]}$ in the Borel construction factors through the map

$$p : (F \times [0, 1]) \times_{A_p} E \rightarrow B \times [0, 1]$$

with the fiber F . We show that p is completely regular. By taking an invariant metric on $F \times [0, 1]$ we may assume that p has isometric fibers over $B \times t$ for every t . By Theorem 6.4 (or by Proposition 6.6) we obtain that p is completely regular over each $B \times t$. Thus, it suffices to prove that p is completely regular over $b \times [0, 1]$ uniformly on $b \in B$ in the following sense: The number $\delta((b, t), \epsilon)$ from the definition of complete regularity can be chosen to be independent of b . For (b, t) and (b, t') we define a homeomorphism $h_{t, t'}$ of the fibers by fixing $e \in E$ with $q_E(e) = b$ and identifying $p^{-1}(b, t)$ by means of the inverse of the projection to the orbit space $q : F \times I \times E \rightarrow (F \times I) \times_{A_p} E$ with $F \times t \times e$ then translating it to $F \times t' \times e$ and projecting by q to $p^{-1}(b, t')$. This homeomorphism does not depend on choice of e . The translation of $F \times t \times e$ to $F \times t' \times e$ is an ϵ -move where ϵ depends on t and $|t - t'|$ only with $\epsilon \rightarrow 0$ as $t' \rightarrow t$. Thus, $h_{t, t'}$ is an ϵ -move for all $b \in B$.

If Question 6.5 has an affirmative answer, then p is a Hurewicz fibration. Then the identification $F \times \{1\} \times B = p^{-1}(B \times \{1\})$ extends to a fiberwise map $F \times [0, 1] \times B \rightarrow (F \times [0, 1]) \times_{A_p} E$ over $B \times [0, 1]$.

The restriction of this fiber wise map over $B \times \{0\}$ yields a splitting of the fiberwise map in the Injectivity Conjecture. \square

6.9. Proposition. *Suppose that $A_p \subset \text{Homeo}(D^n, \partial D^n)$. Then A_p admits a deformation $h_t : A_p \rightarrow \text{Homeo}(D^n, \partial D^n)$ to the identity such h_t is a group homomorphism for every t .*

Proof. This follows from the Alexander's trick. Let tD^n denote the image of D^n under multiplication by $t \leq 1$. Extending to $D^n \setminus tD^n$ by the identity the identity homeomorphism of the boundary ∂tD^n defines an embedding $h_t : \text{Homeo}(tD^n, \partial tD^n) \rightarrow \text{Homeo}(D^n, \partial D^n)$ of topological groups with the image of h_t converging to the unit as $t \rightarrow \infty$. By precomposing this embedding with the given embedding $A_p \rightarrow \text{Homeo}(D^n, \partial D^n)$ and the isomorphism

$$(L_t)_* : \text{Homeo}(D^n, \partial D^n) \rightarrow \text{Homeo}(tD^n, \partial tD^n)$$

where $L_t : D^n \rightarrow tD^n$ is the multiplication by t we obtain a desired deformation. \square

6.10. Corollary. *If every completely regular map with the fiber D^n is a Hurewicz fibration, then the Injectivity Conjecture holds true for any uniformly bounded A_p -action on \mathbb{R}^n .*

We call a G -action on a metric space X *uniformly bounded* if there is an upper bound on the diameter of orbits. Note that an action of A_p on a closed aspherical manifold defines a uniformly bounded action on its universal cover.

Perhaps Edwards-Kirby's theorem would allow to extend Proposition 6.9 to all manifolds. We recall that by the Edwards-Kirby theorem [EK] (which goes back to the proof of Chernavsky's theorem on the local contractibility of $\text{Homeo}(M)$ [Che]) every homotopic to the identity homeomorphism $h : M \rightarrow M$ of a closed manifold can be presented as a finite composition $h = h_n \circ \cdots \circ h_1$ of homeomorphisms fixing the complement to a ball.

7. MODULI SPACE OF TOPOLOGICAL MANIFOLDS

Let F be a compact metric space. We denote by $\text{Emb}(F) = \{\phi : F \rightarrow s\}$ the space of all topological embeddings of F into the pseudo interior $s = (0, 1)^\omega$ of the Hilbert cube $Q = [0, 1]^\omega$ with the sup metric:

$$d(\phi_1, \phi_2) = \sup\{\|\phi_1(x) - \phi_2(x)\| \mid x \in F\}.$$

The pseudo interior is chosen in order to deal with tame embeddings only.

The following theorem implies that the space $Emb(F)$ is an absolute neighborhood extensor for compact metrizable spaces, $ANE(\mathcal{C})$. In particular, it implies that $Emb(F)$ is n -connected and locally n -connected for all n .

7.1. Theorem ([Ch]). *Let (A, A_0) be a compact pair and $f : A \rightarrow s$ is a map. Then for any $\epsilon > 0$ there is an ϵ -close map $g : A \rightarrow s$ that agrees with f on A_0 and is an embedding on $A \setminus A_0$.*

We note that the group of homeomorphisms of F taken with the compact-open topology, $H = Homeo(F)$, acts on $Emb(F)$ from the right by precomposing: $\phi \rightarrow \phi \circ h$, $h \in H$. Thus $\phi_1, \phi_2 \in Emb(F)$ are in the same orbit if and only if $im\phi_1 = im\phi_2$. Note that H acts on $Emb(F)$ by isometries: $d(\phi_1, \phi_2) = d(\phi_1 \circ h, \phi_2 \circ h)$. We call the orbit space of such action *the moduli space* of F and denote it by $\mathcal{M}(F)$. Let $q_F : Emb(F) \rightarrow \mathcal{M}(F) = Emb(F)/H$ be the projection to the orbit space. We consider the quotient metric ρ on $\mathcal{M}(F)$:

$$\rho(\phi_1 H, \phi_2 H) = \inf\{d(\phi_1 \circ h, \phi_2) \mid h \in H\}.$$

We check that ρ is a metric. It is symmetric, since $d(\phi_1 \circ h, \phi_2) = d(\phi_1, \phi_2 \circ h^{-1})$. If $\phi_1 H \neq \phi_2 H$, then $im\phi_1 \neq im\phi_2$. Then for any $h_1, h_2 \in H$,

$$d(\phi_1 \circ h_1, \phi_2 \circ h_2) \geq d_H^Q(im\phi_1, im\phi_2) > 0$$

where d_H^Q is the Hausdorff distance on the closed subsets of Q . Let for $i = 1, 2$ h_i be such that $d(\phi_i h_i, \phi_3) - \rho(\phi_i H, \phi_3 H) < \epsilon/2$. Then the triangle inequality follows when $\epsilon \rightarrow 0$:

$$\begin{aligned} \rho(\phi_1 H, \phi_2 H) &\leq d(\phi_1 h_1, \phi_2 h_2) \leq d(\phi_1 h_1, \phi_3) + d(\phi_2 h_2, \phi_3) < \\ &< \rho(\phi_1 H, \phi_3 H) + \rho(\phi_3 H, \phi_2 H) + \epsilon. \end{aligned}$$

We will identify each orbit $\phi H \in \mathcal{M}(F)$ with the subset $\phi(F)$ of the Hilbert cube.

7.2. Proposition. *For $F_1, F_2 \in \mathcal{M}(F)$, $\rho(F_1, F_2) < \epsilon$ if and only if there is a homeomorphism $g : F_1 \rightarrow F_2$ with the displacement*

$$D_g = \max\{\|g(x) - x\| \mid x \in F_1\} < \epsilon.$$

Proof. Let $F_i = \phi_i(F)$, $\phi_i \in Emb(F)$, $i = 1, 2$.

In one direction, since $d(\phi_1, \phi_2 h) < \epsilon$ for some $h \in H$, we obtain that $D_g < \epsilon$ for $g = \phi_2 h \phi_1^{-1}$.

In the other direction, $d(\phi_1, \phi_2 h) < \epsilon$ for $h = \phi_2^{-1} g \phi_1$ if $D_g < \epsilon$. \square

Let $\mathcal{E}(F) = \{(F', x) \in \mathcal{M}(F) \times s \mid x \in F'\} \subset \mathcal{M}(F) \times s$ and let $\nu_F : \mathcal{E}(F) \rightarrow \mathcal{M}(F)$ be the restriction of the projection onto the first factor. We note that ν_F is completely regular.

7.3. Proposition. *For every continuous map $f : X \rightarrow \mathcal{M}(F)$ the pull-back $f^*(\nu_F)$ is completely regular.*

For every completely regular map $p : X \rightarrow Y$ between compact metric spaces with fiber F there is a continuous map $f : Y \rightarrow \mathcal{M}(F)$ such that $p = f^(\nu_F)$.*

Proof. Any embedding $j : X \rightarrow s$ defines a map $f : Y \rightarrow \mathcal{M}(F)$ by $f(y) = j(p^{-1}(y))$. Let $y_k \rightarrow y$ be a convergent sequence in Y . Then there is a sequence of homeomorphisms $h_k : j(p^{-1}(y)) \rightarrow j(p^{-1}(y_k))$ with $D_{h_k} \rightarrow 0$. By Proposition 7.2 $\rho(f(y), f(y_k)) \rightarrow 0$. Therefore f is continuous. Clearly, p is isomorphic to $f^*(\nu_F)$. \square

7.1. Moduli space of Q -manifolds.

7.4. Proposition. *Let F be such that $\text{Homeo}(F)$ is locally contractible. Then*

(1) $\mathcal{M}(F)$ is locally path connected.

(2) $q_F : \text{Emb}(F) \rightarrow \mathcal{M}(F)$ is a Serre fibration with the fiber $q_F^{-1}(y) \cong \text{Homeo}(F)$ for all $y \in \mathcal{M}(F)$.

Proof. (1) Let F_1 and F_2 be elements of $\mathcal{M}(F)$ at a distance $\rho(F_1, F_2) < \delta$. Thus, $F_1, F_2 \subset s$ and there is a homeomorphism $h : F_1 \rightarrow F_2$ with the displacement $D_h < \delta$. Let $f : F \rightarrow F_1$ be a homeomorphism. We consider a linear homotopy $H : F \times I \rightarrow s$ between f and $h \circ f$. By Theorem 7.1 there is a δ -approximation $H' : F \times I \rightarrow s$ of H by an embedding that coincides with H on $F \times \{0, 1\}$. This defines a path from F_1 to F_2 in $\mathcal{M}(F)$ of diameter $< 2\delta$.

(2) Let $H : I^n \times I \rightarrow \mathcal{M}(F)$ and $h : I^n \times \{0\} \rightarrow \text{Emb}(F)$ with $q_F h = H|_{I^n \times \{0\}}$. In view of Theorem 6.1, $H^*(\nu_F)$ is a trivial fiber bundle with fiber F . The map h defines a trivialization of $H^*(\nu_F)$ over $I^n \times \{0\}$. The projection $I^n \times I \rightarrow I^n$ defines extension of that trivialization to a trivialization over whole $I^n \times I$. This trivialization defines a lift \tilde{H} of H that extends h . \square

We use the standard notation LC^n for the class of locally n -connected spaces.

7.5. Theorem (G.S. Ungar [U]). *Let $p : E \rightarrow B$ be a Serre fibration of metric spaces, E is LC^n , $p^{-1}(b)$ is LC^{n-1} for all $b \in B$ and B is LC^0 . Then B is LC^n .*

7.6. Corollary. *Suppose that $H = \text{Homeo}(F)$ is a locally contractible. Then $\mathcal{M}(F)$ is LC^n for all n .*

Proof. We apply Theorem 7.5 to the map $q_F : \text{Emb}(F) \rightarrow \mathcal{M}(F)$. Since $\text{Emb}(F) \in ANE$ and in view of Proposition 7.4 we obtain that $\mathcal{M}(F)$ is LC^n . \square

We denote by $ANE(n)$ the class of absolute neighborhood extensors for n -dimensional compact metric spaces. Note Kuratowski's theorem characterizes $ANE(n)$ s as LC^n spaces.

7.7. Theorem. *For any compact Q -manifold F , $\mathcal{M}(F) \in ANE(n)$ for all n .*

Proof. We apply Ferry's theorem [F4] which states that $Homeo(F)$ is an ANE for a Q -manifold F , Corollary 7.6, and Kuratowski's characterization of $ANE(n)$. \square

7.8. Problem. *Let F be a compact Q -manifold. Is the space $\mathcal{M}(F)$ an absolute neighborhood extensor for compact metric spaces?*

Since $\mathcal{M}(F)$ is the orbit space of an action by isometries of an ANE group upon an $ANE(\mathcal{C})$ space, it would not be a big surprise that the above problem has a positive answer.

7.9. Theorem. *The affirmative answer to Problem 7.8 implies the Injectivity Conjecture.*

Proof. Assume that the Injectivity Conjecture failed to be true for M . Then there is a compact space B , nonzero $\alpha \in h^*(B)$, a sequence of Hurewicz fibrations $p_k : E_k \rightarrow B$ with isometric fibers M , and a sequence of fiberwise ϵ_k -maps $f_k : E_k \rightarrow M \times B$, $\epsilon \rightarrow 0$ such that $f_k^* \pi_B^*(\alpha) = 0$ for all k . The latter implies that $p_k^*(\alpha) = 0$ for all k . Here π_B and π_M denote projections of the product $B \times M$ to the factors.

We define a compactification X of $\coprod E_k$ by M as the subspace

$$X = \coprod_k G_k \cup \{a\} \times M \subset \alpha(\coprod_k E_k) \times M$$

of the product of the one-point compactification of the union of E_k and M where $G_k \subset E_k \times M$ is the graph of the composition $\pi_M \circ f_k$ and a is the compactifying point in $\alpha(\coprod_k E_k)$. The Q -manifold versions of Theorem 6.4 and Theorem 6.2 imply that the union of p_k defines a completely regular map $p : X \rightarrow \alpha(\coprod_k B_k)$ with fibers M where each B_k is homeomorphic to B . By Proposition 7.3, $p = f^*(\nu_M)$ for some continuous map $f : \alpha(\coprod_k B_k) \rightarrow \mathcal{M}(M)$.

We present $B = \lim_{\leftarrow} \{L_i, \phi_j^i\}$ as the limit of an inverse sequence of compact polyhedra. Let T be the natural compactification of $\coprod L_i$ by B . If $\mathcal{M}(M)$ is an ANE for compact metric spaces, then there is an extension $\bar{f} : W \rightarrow \mathcal{M}(M)$ to a neighborhood of $\alpha(\coprod_k B_k)$ in $\alpha(\coprod_k T_k)$. We note that there is k such that $T_k \subset W$ and the restriction of \bar{f} to T_k is null-homotopic. Let $\bar{p}_k : Z_k \rightarrow T_k$ be the restriction of $\bar{f}^*(\nu_M)$ over T_k .

For sufficiently large i , there is an $\alpha_i \in h^*(L_i)$ that maps to $\alpha \in h^*(B)$. Let $U_i \subset T$ be the compactification of $L_i \amalg L_{i+1} \amalg \dots$ by B . The bonding map $B \rightarrow L_i$ factors through U_i ; let $\alpha'_i \in h^*(U_i)$ be the image of α_i . Let $V_i \subset Z_k$ be the preimage in Z_k of the copy of U_i in T_k . Let $\beta'_i \in h^*(V_i)$ be the image of α'_i . Since α'_i maps to α , the image of β'_i in $h^*(E_k)$ is the same as the image of α , which is trivial by the assumption. Hence the image of β'_i in $h^*(V_j)$ is trivial for some $j > i$. If $\alpha'_j \in h^*(U_j)$ is the image of α'_i , then the image of α'_j in $h^*(V_j)$ is trivial. Finally, if $\alpha_j \in h^*(L_j)$ is the image of α_i , then α_j maps to α'_j under the map $U_j \rightarrow L_j$, which in turn goes to 0 under the restriction $V_j \rightarrow U_j$ of \bar{p}_k . The composition $(\bar{p}_k)^{-1}(L_j) \rightarrow V_j \rightarrow U_j \rightarrow L_j$ coincides with the restriction $(\bar{p}_k)^{-1}(L_j) \rightarrow L_j$ of \bar{p}_k , and therefore α_j goes to zero under the latter restriction. This contradicts the fact that p_k is a trivial bundle over L_j . \square

REFERENCES

- [A] P. S. Alexandroff, *Dimension theorie, ein Betrag zur Geometrie der abgeschlossen Mengen* Math. Ann. 106 (1932), 161-238.
- [AS] M. Atiyah and G. Segal, *Equivariant K-theory and completion*. J. Differential Geometry 3 (1969) 1-18.
- [Ba] T. Bartsch *On the genus of representation spheres*, Comment. Math. Helvetici, Vol 65 (1990), 85-95.
- [AB] A. Borel, *Seminar on transformation groups*, Annals of mathematical Studies, vol 46, Princeton University Press, 1960.
- [Bo] K. Borsuk, *Theory of retracts*, PWN, Warszawa 1967.
- [BRW] G. E. Bredon, F. Raymond, and R. F. Williams, *p-Adic Groups of transformations*, Trans. AMS, vol 99, No 3 (1961), 488-498.
- [Ch] T. A. Chapman, *Lectures on Hilbert Cube manifolds*, CBMS No 28, AMS, Providence, 1975.
- [CF] T. A. Chapman and S. Ferry, *Hurewicz fiber maps with ANR fibers*. Topology 16 (1977), no. 2, 131-143.
- [CF2] T. A. Chapman, S. Ferry *Approximating homotopy equivalences by homeomorphisms*. Amer. J. Math. 101 (1979), no. 3, 583-607.
- [Che] A. V. Chernavsky, *Local contractibility of the group of homeomorphisms of a manifold*. Mat. Sb. (N.S.) 79 (121) (1969) 307-356.
- [CLOT] O. Cornea, G. Lupton, J. Oprea, D. Tanré, *Lusternik-Schnirelmann Category*, Mathematical Surveys and Monographs, vol. 103, AMS, 2003.
- [Dr] A. Dranishnikov, *The LS-category of the product of lens spaces*, AGT, to appear.
- [DFW] A. N. Dranishnikov S. C. Ferry, S. Weinberger, *Large Riemannian manifolds which are flexible*. Ann. of Math. (2) 157 (2003), no. 3, 919-938.
- [DS] A. N. Dranishnikov and E.V. Schepin, *Cell-like mappings. The problem of the increase of dimension*. (Russian) Uspekhi Mat. Nauk 41 (1986), no. 6(252), 49-90.

- [DW] A. N. Dranishnikov and J. E. West, *Correction to: "Compact group actions that raise dimension to infinity"* [Topology Appl. 80 (1997), no. 1-2, 101-114; MR1469471]. Topology Appl. 135 (2004), no. 1-3, 249-252.
- [DH] E. Dyer and M.-E. Hamstrom, *Completely regular mappings*, Fund. Math. 45 (1958), 103-118.
- [EK] R. D. Edwards and R. C. Kirby, *Deformation of spaces of embeddings*. Ann. of Math. (2) 93 (1971), 63-88.
- [F1] S. Ferry, *A stable converse to the Vietoris-Smale theorem with applications to shape theory*. Trans. Amer. Math. Soc. 261 (1980), no. 2, 369-386.
- [F2] S. Ferry, *Strongly regular mappings with compact ANR fibers are Hurewicz fiberings*. Pacific J. Math. 75 (1978), no. 2, 373-382.
- [F3] S. Ferry, *Homotoping ϵ -maps to homeomorphisms*. Amer. J. Math. 101 (1979), no. 3, 567-582.
- [F4] S. Ferry, *The homeomorphism group of a compact Hilbert cube manifold is an ANR*. Ann. Math. (2) 106 (1977), no. 1, 101-119.
- [G] A. Gleason, *Groups without small subgroups*, Annals of Math. 56 (1952), 193-212.
- [Ha] M. E. Hamstrom, *Completely regular mappings whose inverses have LC^0 homeomorphism group: A correction*. 1971 Proc. First Conf. on Monotone Mappings and Open Mappings (SUNY at Binghamton, Binghamton, N.Y., 1970) pp. 255-260 State Univ. of New York at Binghamton, Binghamton, N.Y.
- [H] D. Hilbert, *Mathematical Problems*, Bul. Amer. Math. Soc. 8 (1901-02), 437-479.
- [Ka] T. Kambe, *The structure of K_Λ -rings of the lens space and their applications*, J. Math. Soc. Japan, Vol. 18, No 2, 1966, 135-146.
- [KS] T. Kawaguchi and M. Sugawara, *K - and KO -Rings of the lens space $L^n(p^2)$ for odd prime p* , Hiroshima Math. J. 1 (1971), 273-286.
- [K] Kim, Soon-Kyu, *Local triviality of completely regular mappings*. Duke Math. J. 38 (1971), 467-471.
- [La] R.C. Lacher, *Cell-like mappings and their generalizations*, Bull. Amer. Math. Soc.(2), 83 No 4, 1977, 495-552.
- [Le] M. Levin, *Resolving rational cohomological dimension via a Cantor group action*, AGT, to appear.
- [Ma] N. Mahammed, *A propos de la K -theorie des espaces lenticulaires*, C. R. Acad. Sci. Paris, 271 (1970), 639-642.
- [Mal] I. Maleshich, *The Hilbert-Smith conjecture for Hölder actions*, Russian Math. Surveys 52 (1997), no. 2, 407-408.
- [Mar] G. J. Martin, *The Hilbert-Smith conjecture for quasiconformal actions*, Electron. Res. Announc. Amer. Math. Soc. 5 (1999), 66-70 (electronic).
- [Me] D. Meyer, *\mathbb{Z}/p -equivariant maps between lens spaces and spheres*, Math. Ann. 312, (1998) 197-214.
- [M] E. Michael, *Continuous selections. III*, Ann. of Math. (2) 65 (1957), 375-390.
- [MZ1] D. Montgomery and L. Zippin, *Small groups of finite dimensional groups*, Annals of Math. 56 (1952), 213-241.
- [MZ2] D. Montgomery and L. Zippin, *Topological Transformation Groups*, Wiley, N.Y., 1955.

- [N] M. H. A. Newman, *A theorem on periodic transformation spaces*, Q. Journ. Math. 2 (1931), 1-9.
- [Pa] J. Pardon, *The Hilbert-Smith conjecture for three-manifolds*, J. Amer. Math. Soc. 26 (2013), no. 3, 879-899.
- [P] L. S. Pontryagin, *Topological groups*, Princeton University Press, Princeton, 1939.
- [R] F. Raymond, *Cohomological and Dimension Theoretical properties of Orbit Spaces of p -Adic actions*, Proceedings of the Conference on Transformation Groups, New Orleans, 1967, Springer 1968, 354-365.
- [RW] F. Raymond and R. F. Williams, *Examples of p -adic transformation groups*, Ann. of Math. (2) 78 (1963), 92-106.
- [RSc] D. Repovs and E. V. Schepin, *A proof of the Hilbert-Smith conjecture for actions by Lipschitz maps*, Math. Ann. 308 (1997), no. 2, 361-364.
- [RS] D. Repovs and P. V. Semenov, *Continuous Selection of Multivalued Mappings*, Kluwer, 1998.
- [Ru] Y. B. Rudyak, *On category weight and its applications*. Topology 38 (1999), no. 1, 37-55.
- [Sch] A. S. Schwarz, *The genus of a fibered space*. Amer. Math. Sci. Transl. 55 (1966), 49-140.
- [Se] S. B. Seidman, *Completely regular mappings with locally compact fiber*. Trans. Amer. Math. Soc. 147 (1970), 461-471.
- [Sm] P. A. Smith, *Transformations of finite period.III Newman's Theorem*, Ann. Math. 42 (1941), 446-457.
- [Sm2] P. A. Smith, *Periodic and nearly periodic transformations*, Lectures in Topology, Univ. of Michigan Press, Ann Arbor, 1941.
- [Sr] T. Srinivasan, *On the Lusternik-Schnirelmann category of Peano continua*, Top. Appl. 160 (2013), no 13, 1742-1749.
- [Sw] R.M. Switzer, *Algebraic Topology - Homotopy and Homology*, Springer, 1975.
- [U] G. S. Ungar, *Completely regular maps, fiber maps and local n -connectivity*. Proc. Amer. Math. Soc. 21 (1969) 104-108.
- [V] J. W. Vick *An application of K -theory to equivariant maps*, Bull. Amer. Math. Soc. 75 1969, 1017-1019.
- [Wa] J. Walsh, *Dimension, cohomological dimension, and cell-like mappings*, Lecture Notes Math. 870 105-118.
- [Wi] R. F. Williams, *The construction of certain 0-dimensional groups*, Trans. Amer. Math. Soc. 129 (1967), 140-156.
- [Y1] C. T. Yang, *p -Adic transformation groups* Mich. Math. J. 7 (1960), 201-218.
- [Y2] C. T. Yang, *Hilbert's Fifth Problem and related problems on transformation groups*, Proceedings of Symposia in Pure Mathematics; Mathematical development arising from Hilbert Problems 1974, edited by F. Browder, XXVIII, Part I, 142-164.

ALEXANDER N. DRANISHNIKOV, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, 358 LITTLE HALL, GAINESVILLE, FL 32611-8105, USA
E-mail address: dranish@math.ufl.edu